International Journal of Theoretical Physics, Vol. 35, No. 7, 1996

Nonclassical Representation of $osp_q(1/2)$ Algebra and Completeness Relation of q-Deformed Supercoherent States

Won-Sang Chung[1]

Received September 27, 1995

In this paper some nonclassical representations of $osp_q(1/2)$ algebra are presented and the completeness relation of q-deformed supercoherent states is proved.

1. INTRODUCTION

Quantum groups or the q-deformed Lie algebra implies some specific deformation of a classical Lie algebra. From a mathematical point of view, it is a noncommutative associative Hopf algebra. After Jimbo (1985, 1986) and Drinfeld (1986) introduced the q-deformed $su(2)$ algebra [$su_q(2)$ algebra], Kulish (1988) and Saleur (1990) showed that a q-deformation of the graded algebra $osp(1/2)$ algebra could also be defined, in relation to the graded Yang–Baxter equation (Kulish and Sklyanin, 1982).

In this paper we present a nonclassical representation of $osp_q(1/2)$ algebra and prove the completeness relation for the q-spin coherent state. This representation is not defined in the classical limit $q \rightarrow 1$, but it enables us to obtain the completeness relation for the q-spin supercoherent states of $osp_q(1/2)$ algebra.

2. NONCLASSICAL REPRESENTATION OF $osp_q(1/2)$ ALGEBRA

In this section we present a nonclassical realization of $osp_q(1/2)$ algebra. Consider the $osp_q(1/2)$ algebra given by

$$[H, v_{\pm}] = \pm v_{\pm}, \qquad \{v_{+}, v_{-}\} = [2H] \tag{1}$$

[1] Theory Group, Department of Physics, College of Natural Sciences, Gyeongsang National University, Jinju, 660-701, Korea.

0020-7748/96/0700-1289$09.50/0

where the q-number $[x]$ is defined as

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$$

We introduce the bases

$$v_+ e_n = f(n) e_{n+1}, \qquad v_- e_n = g(n) e_{n-1}$$
$$He_n = (n - 2j) e_n, \qquad n = 0, 1, 2, \ldots, 4j \tag{2}$$

where we assumed that there exists a ground state e_0 satisfying

$$v_- e_0 = 0 \tag{3}$$

This representation is $4j + 1$ dimensional. There exist $2j + 1$ even states (bosonic states) and $2j$ odd states (fermionic states). Here we assumed that j is integer or half odd integer.

From the $osp_q(1/2)$ algebra we obtain

$$\begin{aligned} f(n-1)g(n) &= \sum_{k=0}^{n-1} (-)^{n-1-k} [-4j + 2k] \\ &= (-)^{n-1} \frac{q + q^{-1}}{q - q^{-1}} [n]_+ [4j - n + 1]_+ \end{aligned} \tag{4}$$

where the q-fermionic number $[x]_+$ is defined as

$$[x]_+ = \frac{q^{-x} - (-)^x q^x}{q + q^{-1}} \tag{5}$$

When q goes to 1, the right-hand side of equation (4) diverges, so we cannot determine the functions $f(n)$ and $g(n)$. Thus this representation does not have a classical analogue. From now on we assume that $q \neq 1$.

The first choice for the representation is

$$v_+ e_n = (-1)^{n/2} \left(\frac{q + q^{-1}}{q - q^{-1}} \right)^{1/2} ([n+1]_+ [4j - n]_+)^{1/2} e_{n+1}$$
$$v_- e_n = -(-)^{(n-1)/2} \left(\frac{q + q^{-1}}{q - q^{-1}} \right)^{1/2} ([n]_+ [4j - n + 1]_+)^{1/2} e_{n-1}$$
$$He_n = (n - 2j) e_n, \qquad n = 0, 1, \ldots, 4j \tag{6}$$

The second choice for the representation is obtained by replacing n by $2j + m$,

$$v_+ e_m = (-)^{(2j+m)/2}\left(\frac{q+q^{-1}}{q-q^{-1}}\right)^{1/2}([2j+m+1]_+[2j-m]_+)^{1/2}e_{m+1}$$

$$v_- e_m = -(-)^{(2j+m-1)/2}\left(\frac{q+q^{-1}}{q-q^{-1}}\right)^{1/2}([2j+m]_+[2j-m+1]_+)^{1/2}e_{m-1}$$

$$He_m = me_m, \qquad m = -2j, -2j+1, \ldots, 2j \tag{7}$$

On this representation space we have $v_\pm^* = -v_+$ and $H^* = H$. This representation $D(4h)$ is defined on the $(4j + 1)$-dimensional Hilbert space H_{4j} with orthonormal basis $\{e_m: n = -2j, \ldots, 2j\}$ such that

$$\langle e_m, e_{m'}\rangle = \delta_{mm'} \tag{8}$$

The third convenient basis for H_{4j} is the set $\{f_n: n = 0, 1, \ldots, 4j\}$ such that

$$v_+ f_n = \left(\frac{q+q^{-1}}{q-q^{-1}}\right)^{1/2}[4j-n]_+ f_{n+1}$$

$$v_- f_n = (-)^{n-1}\left(\frac{q+q^{-1}}{q-q^{-1}}\right)^{1/2}[n]_+ f_{n-1}$$

$$Hf_n = (n-2j)f_n, \qquad n = 0, 1, \ldots, 4j \tag{9}$$

Let us introduce a convenient one-variable model of $D(4h)$. Here the vector space H_{4j} consists of polynomials $f(z)$ of maximum order $4j$ in the complex variable z. The action of the $osp_q(1/2)$ algebra is defined by the operators

$$v_+ = (q^2 - q^{-2})^{-1}z(q^{-4j}T_z - q^{4j}RT_z^{-1})$$

$$v_- = (q^2 - q^{-2})^{-1}\frac{1}{z}(T_z - RT_z^{-1})$$

$$H = -2j + z\frac{d}{dz} \tag{10}$$

where

$$T_z^\alpha f(z) = f(q^\alpha z), \qquad Rf(z) = f(-z)$$

From the above representation we have

$$v_- f_0 = 0, \qquad v_+ f_{4j} = 0 \tag{11}$$

3. COMPLETENESS RELATION OF q-SPIN SUPERCOHERENT STATES

In this section we prove the completeness relation of q-spin supercoherent states of the $osp_q(1/2)$ algebra. In order to do so it is necessary to investigate some properties of fermionic q-numbers.

The fermionic q-number $[x]_+$ has peculiar properties. For example, in the limit $q \to 1$, we have

$$[x]_+ \to \frac{1 - (-)^x}{2} \tag{12}$$

We can construct the fermionic q-derivative as follows:

$$D_+ f(x) = \frac{f(q^{-1}x) - f(-qx)}{x(q + q^{-1})} \tag{13}$$

Then we obtain

$$D_+ x^n = [n]_+ x^{n-1} \tag{14}$$

The fermionic q-deformed exponential function

$$e_q(x) = \sum_{n=0}^{\infty} \frac{1}{[n]_+!} x^n \tag{15}$$

satisfies

$$D_+ e_q(x) = e_q(x) \tag{16}$$

Its inverse operator, called the fermionic q-integral, is defined as

$$\int_0^x d_+x\, F(x) = (q + q^{-1})x \sum_{n=0}^{\infty} (-q)^n q^{n+1} F((-q)^n q^{n+1} x) \tag{17}$$

Then we have

$$\int_0^x d_+x\, x^n = \frac{x^{n+1}}{[n+1]_+} \tag{18}$$

and

$$\int_0^x d_+x\, e_q(x) = e_q(x) \tag{19}$$

From the definition of fermionic q-derivative we obtain the deformed Leibniz rule for the fermionic q-derivative

$$D_+(f(x)g(x)) = f(q^{-1}x)D_+g(x) + g(-qz)D_+f(x) \tag{20}$$

or

$$D_+(f(x)g(x)) = f(-qx)D_+g(x) + g(q^{-1}x)D_+f(x) \tag{21}$$

From the definition of fermionic q-integration, we obtain the fermionic q-integration-by-parts formula:

$$\int_0^x d_+x\, f(q^{-1}x)D_+g(x) = [f(x)g(x)]_0^x - \int_0^x d_+x\; g(-qx)D_+f(x) \tag{22}$$

or

$$\int_0^x d_+x\, f(-qx)D_+g(x) = [f(x)g(x)]_0^x - \int_0^x d_+x\; g(q^{-1}x)D_+f(x) \tag{23}$$

In Section 2 we presented three types of representation of $osp_q(1/2)$ algebra. In order to prove the completeness relation for q-deformed supercoherent states, we should introduce the new representation of the algebra (1). Let us assume that v_+ is not the (anti) Hermitian conjugate of v_-. Instead we introduce the operator satisfying

$$H'e_n = ne_n \tag{24}$$

Let us assume that

$$v_\mp^* = v_-(-)^{H'}, \qquad v_\pm^* = (-)^{H'}v_+ \tag{25}$$

where $v_\pm^*$ means the dual of $v_\pm$. Then the algebra (1) remains invariant after acting with the * operation.

From the conjugate relation

$$\langle e_n, v_-e_{n+1}\rangle = \langle v_\pm^* e_n, e_{n+1}\rangle \tag{26}$$

we get the relation $g(n) = (-)^n f(n - 1)$ in Eq. (4). Using this, we have the representation

$$v_+e_n = i(-)^{n+1}\left(\frac{q + q^{-1}}{q - q^{-1}}\right)^{1/2}([n + 1]_+[4j - n]_+)^{1/2}e_{n+1}$$

$$v_-e_n = i\left(\frac{q + q^{-1}}{q - q^{-1}}\right)^{1/2}([n]_+[4j - n + 1]_+)^{1/2}e_{n-1}$$

$$He_n = (n - 2j)e_n, \qquad n = 0, 1, \ldots, 4j \tag{27}$$

Then the q-spin coherent state $|z\rangle$ is defined as

$$|z\rangle = e_q\left(\left(\frac{q - q^{-1}}{q + q^{-1}}\right)^{1/2} zv_-\right)e_{4j} \tag{28}$$

where e_{4j} is highest state vector for representation (27). Then we get

$$|z\rangle = \sum_{n=0}^{4j} i^n z^n \left(\frac{[4j]_+!}{[n]_+![4j - n]_+!}\right)^{1/2} e_{4j-n} \tag{29}$$

Then this q-spin supercoherent state satisfies the completeness relation

$$\int d_+|z|^2\, d\theta\, |z\rangle\langle z|\, \mu(|z|^2) = 1 \tag{30}$$

where

$$\mu(|z|^2) = \frac{1}{2\pi}[4j + 1]_+(1 + |z|^2)^{-4j-2} \tag{31}$$

In deriving Eq. (30) we used the formula

$$\int d_+x\, x^n(1 + x)^{-m} \tag{32}$$

$$= (-)^{n(m+1)}(-)^m \frac{[n]_+!\,[m - n - 2]_+!}{[m - 1]_+!} \qquad (n < m)$$

4. CONCLUSION

In this paper we obtained some nonclassical representations of $osp_q(1/2)$ algebra and used them to prove the completeness relation for q-spin supercoherent states. This representation has no classical analogue. It is not defined in the limit $q \to 1$. At this stage we have an open problem: Is it possible to define the q-deformed supercoherent state so that it satisfies the completeness relation and can be defined in the classical limit?

ACKNOWLEDGMENTS

This work was supported by the Non Directed Research Fund, Korea Research Foundation, 1994, and by KOSEF through C.T.P. at Seoul National University, as well as by the Basic Science Research Program, Ministry of Education, 1994 (BSRI-94-2413).

REFERENCES

Drinfeld, V. (1986). In *Proceedings of the International Congress of Mathematicians*, Berkeley, p. 78.
Jimbo. (1985). *Letters in Mathematical Physics*, **10**, 63.
Jimbo. (1986). *Letters in Mathematical Physics*, **11**, 247.
Kulish, P. P. (1988). Quantum superalgebra *osp*(1, 2), Preprint.
Kulish, P. P., and Sklyanin, E. K. (1982). *Journal of Soviet Mathematics*, **19**, 1596.
Saleur, H. (1990). *Nuclear Physics B*, **336**, 363.